Newly discovered Potentials for Topos Theory from Grothendieck's Handwritten Notes: Functorial Correspondences and Topos Duality: A Reconstruction from Pages (Cote 115)

Peter de Ceuster , in dedication of Maryam Mirzakhani†

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Abstract

This abstract delivers a remark related to Topos Duality. We base ourselves on Alexander Grothendieck's handwritten manuscript *[from 1982] 103 Functorial "correspondences". Duality of topos: handwritten notes (nd). Rating No. 115 (14 p.)*, preserved at the Université de Montpellier archives (see `https://grothendieck.umontpellier.fr/archives-grothendieck/#`). We create our case directly from these 14 page scans, included in this abstract; we will use arrows, 2-cells and triangles that are present in the handwriting and we will transcribe them into bicategorical diagrams and formalize them in the ∞-categorical language accepted today.

Bridging logics and geometry we believe our abstract can help advance Topos theory, through a deeper understanding of modern categorical logic, we regard a topos as the semantics of the theory. Duality then trades geometric morphisms for theory morphisms. In practice this informs how we internalize our construct (e.g. synthetic algebraic geometry, cohesive/synthetic homotopy theory). Please note, Grothendieck through his notes, seems to sketch the clear functoriality required for such a bridge.

Topos-theoretic Galois theory. Dualities between atomic/Boolean topoi and profinite group actions inspire contemporary refinements: étale homotopy types, profinite or condensed avatars, and generalized Galois categories. The handwritten notes do indicate when a topos is governed by a "Galois object," which we reinterpret to study fundamental groupoids in ∞-topoi and stratified settings.

His notes also appear to cover Morita equivalence for theories.(Two sites can present the same topos; two theories can be Morita-equivalent.) Grothendieck his notes emphasize the principle that equivalence of topoi, not presentation, is the relevant invariant for structural claims. We build further upon said angle, and provide new structural claims.

Contents

1 Introduction and provenance

Our abstract presented is an explicit mathematical rendering, analysis and contemplation of Grothendieck's 14 handwritten pages labeled Cote 115 ("Autour de Pursuing Stacks: Functorial correspondences; Duality of topos"). The scans provided by the university were used directly: arrows and 2-cells were transcribed into Corr-diagrams; marginal notes indicating "correspondance" vs "morphisme" were interpreted as intentional preference for spans/profunctors. The appendix includes the scanned images and a short transcription note for each page.

Our goal is not to perform hagiography but to produce usable mathematics: precise definitions, bicategorical constructions, concrete criteria for when site-functors induce geometric morphisms and when those morphisms admit adjoints, a minimal six-functor formalism derived from the notes, and verification in three contexts: étale ∞-topos, classifying ∞-topos, and condensed set topos. We believe re-framing Grothendieck his mathematics, in a modern context can help Topos theory.

Summary of main contributions

- A precise definition of the bicategory $\mathrm{Corr}(\mathcal{C})$ consistent with Grothendieck's drawn spans and 2-cells.
- Formalization of duality principles Grothendieck annotated: (subtopos $\leftrightarrow$ local operator), (presentation $\leftrightarrow$ theory), (site morphism $\leftrightarrow$ geometric morphism).
- A minimal six-functor axiomatic package for a class $\mathcal{M}$ of "good morphisms," extracted from the handwritten criteria.

- Concrete propositions translating Grothendieck's scribbled Beck–Chevalley diagrams into modern Beck–Chevalley conditions in ∞-topoi.
- Heuristic verifications on three exemplar categories (étale -topos, classifying -topos, condensed set topos).
- Appendix with some of the scanned pages and brief line-by-line Remark.

2 Preliminaries and notation

We work in a modern ∞-categorical context when necessary; when statements are purely 1-categorical we remark so. Let $\mathcal{S}$ denote spaces (-groupoids). For an ∞-category $\mathcal{C}$, we write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the mapping ∞-category. For a topos $\mathcal{E}$ we use $\Omega_{\mathcal{E}}$ for its subobject classifier.

3 Correspondences and their bicategorical structure

Grothendieck's handwritten diagrams repeatedly replace a map $f : X \to Y$ by a "correspondance" or span $X \leftarrow M \to Y$; arrows between spans, small triangles marked as "adj." or with annotations suggesting existence of adjoints, appear throughout. We formalize.

Definition 3.1. *Let $\mathcal{C}$ be an ∞-category admitting finite limits. Define* $\mathrm{Corr}(\mathcal{C})$ *as the bicategory (or $(\infty, 2)$-category) whose objects are objects of $\mathcal{C}$, whose 1-morphisms* $\mathrm{Hom}_{\mathrm{Corr}}(X, Y)$ *are spans $X \xleftarrow{p} M \xrightarrow{q} Y$ (equivalence classes up to equivalence in the middle object), and whose 2-morphisms are maps of spans (i.e. maps $h : M \to M'$ commuting with the leg maps, up to homotopy). Composition is induced by homotopy fiber product in $\mathcal{C}$:*

$$(X \xleftarrow{p} M \xrightarrow{q} Y) \circ (Y \xleftarrow{p'} N \xrightarrow{q'} Z) \;=\; (X \xleftarrow{p \circ \mathrm{pr}_1} M \times_Y N \xrightarrow{q' \circ \mathrm{pr}_2} Z).$$

Proposition 3.2. *If $\mathcal{C}$ is an ∞-topos (or more generally an ∞-category with homotopy pullbacks and suitable size conditions) then* $\mathrm{Corr}(\mathcal{C})$ *is well-defined and composition is associative up to coherent equivalence.*

Sketch. Associativity follows from the universal property of homotopy pullbacks and the fact that homotopy pullbacks are associative up to canonical equivalence; coherence is handled by standard unitality/coherence results for bicategories in ∞-categorical contexts (see Lurie). □

3.1 Adjoints and Beck–Chevalley via spans

Grothendieck's sketches repeatedly show "triangles" labelled with arrows that we transcribe as candidate adjunctions. The notes suggest that a morphism that can be represented by a span with a map having left adjoint should be seen as having $f_!$ etc.; similarly Beck–Chevalley squares are drawn repeatedly.

Definition 3.3. *A span $X \xleftarrow{p} M \xrightarrow{q} Y$ is* left-compactible *(resp.* right-compactible*) if p admits a left adjoint $p_!$ (resp. q admits a right adjoint q_*) when interpreted in the appropriate presheaf or sheaf context (i.e. after sheafifying or passing to ∞-categories of bundles).*

This captures Grothendieck's repeated marginal note: "il faut p adjointable$'$".

Proposition 3.4 (Span-induced push-pull)**.** *Let $\mathcal{C}$ be an ∞-topos. Given a span $X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, there is an induced functor between slice topoi (or sheaf categories)*

$$q_* p^* : (X) \longrightarrow (Y)$$

when the relevant adjoints exist at the presheaf/sheaf level. Composition of spans corresponds to composition of these induced push-pull functors, and Beck–Chevalley conditions translate to natural isomorphisms when the fiber product squares are appropriately adjointable.

Sketch. This is the usual push-pull construction: p^* is precomposition along p, while q_* is right Kan extension along q when defined; sheafification ensures the image remains in sheaves. The details are routine but require checking preservation of covering sieves and compatibility with sheaf conditions. Grothendieck's notes emphasize checking this at the level of correspondences rather than maps, which is precisely why one formulates these as spans. □

4 Duality of topoi and local operators

Grothendieck's pages frequently annotate subtopoi and make arrows to a circled Ω (subobject classifier), suggesting he was thinking in terms of Lawvere–Tierney topology operators. We formalize that perspective and push it into modal/connected contexts.

Theorem 4.1 (Lawvere–Tierney duality, recapitulated)**.** *For an elementary topos $\mathcal{E}$, there is a bijection between subtopoi of $\mathcal{E}$ and Lawvere–Tierney topologies $j : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$ (idempotent, left-exact monads on Ω).*

Remark 4.2. *Grothendieck's marginalia stresses interpreting such local operators as "modalities" in the internal language; we adopt this language and extend to ∞-topoi by considering idempotent left-exact localizations (reflective subcategories) and their associated modalities.*

4.1 Subtopoi as modalities in ∞-topoi

Let $\mathcal{E}$ be an ∞-topos and $L : \mathcal{E} \to \mathcal{E}$ a left-exact localization with fully faithful right adjoint i. Then the essential image of i is a subtopos; the localization L is an ∞-modal operator. This is exactly the ∞-analogue of Lawvere–Tierney topologies and is what Grothendieck's diagrams presage.

5 Criteria: when a functor of sites yields a geometric morphism

Grothendieck wrote several entries giving conditions on a functor $u : (C, J_C) \to (D, J_D)$ so that it induces a morphism of topoi. From the notes we extract and formalize the precise conditions.

Proposition 5.1 (Grothendieck-extracted criterion)**.** *Let $u : (C, J_C) \to (D, J_D)$ be a functor of small sites. If*

1. *u is continuous (preimage of covering sieve is covering),*
2. *u is left-exact (preserves finite limits),*

then u induces a geometric morphism of topoi

$$f : (C, J_C) \longrightarrow (D, J_D)$$

with inverse image f^* given by $f^*(F) = F \circ u$ for $F \in (D, J_D)$.

Remark 5.2. *Grothendieck often wrote additional ad-hoc conditions in the margins when he wanted $f_!$ to exist (left Kan extension along u must preserve sheaves). We record these as additional hypotheses for the existence of $f_!$ or f_* in later statements.*

6 Minimal six-functor formalism extracted from the notes

The manuscript marks certain morphisms as "bons pour correspondances" (good for correspondences): proper-like, finite-type, or those admitting compactifying factorizations. We axiomatically extract a minimal six-functor formalism for a class $\mathcal{M}$ of morphisms in a geometric context.

Definition 6.1 (Minimal six-functor axioms)**.** *Let $\mathcal{M}$ be a class of geometric morphisms between ∞-topoi satisfying:*

1. *closed under composition and base-change;*
2. *for each $f \in \mathcal{M}$ the functors f^*, f_* exist (and $f_!$ exists when f is additionally proper-like in the extracted sense);*
3. *Beck–Chevalley holds for Cartesian squares with maps in $\mathcal{M}$ under the adjointability conditions extracted from the notes;*
4. *projection formula holds for $f \in \mathcal{M}$ under finite presentability hypotheses on coefficients.*

Then $\mathcal{M}$ supports a minimal six-functor formalism.

Proposition 6.2 (Projection and base-change from spans)**.** *Under the axiom scheme above, every span representing a morphism in $\mathcal{M}$ yields push-pull functors on sheaves whose compatibility is controlled by the Beck–Chevalley conditions; hence base-change and projection formulas are encoded at the level of* Corr*.*

Sketch. This is bookkeeping: the push/pull associated to spans compose along fiber products and satisfy the usual interchange laws when the relevant adjoints exist. Grothendieck's diagrams underline precisely these interchange squares. This method of projection might be of interest in the future. □

7 Toposic Tannaka reconstruction derived directly from the notes

Identifying futher points of interest, we come across one theme in the manuscript and that is recognizing a topos from functorial data, a Tannakian flavor. We sketch an ∞-categorical Tannaka-like statement adapted to the notes.

Definition 7.1. *Let $\mathcal{E}$ be a topos and let $\mathscr{C}$ be a symmetric monoidal presentable ∞-category (examples: spectra, derived vector spaces). A* fiber functor *is a symmetric monoidal left-exact accessible functor $\omega : \mathcal{E} \to \mathscr{C}$ that detects enough objects (satisfies a conservative condition).*

Proposition 7.2 (Tannaka-type reconstruction, heuristic)**.** *Under finiteness and rigidity hypotheses (adapted to the particular topos), the automorphism group-object $\mathrm{Aut}^{\otimes}(\omega)$ can be recovered and represents the group-like object controlling $\mathcal{E}$ in analogy with classical Tannaka duality. Grothendieck's notes indicate precisely the data one should try to extract: a category of "representations" and a fiber functor.*

Remark 7.3. *Even though the handwritten pages contain helpful but heuristic cues about the necessary rigidity and finiteness conditions, we were unable to create a full theorum due to the limited pages provided by Grothendieck. Note that the exact hypotheses depend on whether one works in spectral/derived/condensed contexts. Nonetheless these are handy and help depict the overall idea.*

8 From sites to logic: internal languages and modalities

Grothendieck marks subtopoi and local operators, inviting an interpretation as internal modalities. We make the bridge explicit and point to applications in cohesive/synthetic homotopy theory.

Proposition 8.1. *Given a left-exact localization $L : \mathcal{E} \to \mathcal{E}$ (a modality), there is an induced modal connective in the internal language of $\mathcal{E}$, which corresponds to the subtopos* $\mathrm{Im}(i)$ *where i is the right adjoint. The handwriting's repeated mapping from subtopoi to logical marks matches this identification. We find this method appealing enough to proceed into covering echoes.*

9 Galois and anabelian echoes

Grothendieck's notes contain a few explicit sketches of "groupoids of points" and remarks about reconstructing fundamental group(oid) data. We formalize the gist. These echoes are sketches, and provide deep insight into topos theory.

Definition 9.1. *Let $\mathcal{E}$ be a topos and* $\mathrm{Pt}(\mathcal{E})$ *its groupoid of points. The profinite completion of* $\mathrm{Pt}(\mathcal{E})$ *(with relevant topology) is the* Galois object *controlling atomic parts of $\mathcal{E}$ in the sense of Grothendieck's classical Galois theory.*

Proposition 9.2. *If $\mathcal{E}$ is atomic and Boolean, then it is equivalent to the classifying topos of a profinite group action; the handwritten notes sketch this in an anodyne way. In ∞-contexts, one replaces profinite groups with pro-∞-groupoids and checks the same structural correspondences.*

10 Test categories and homotopy

References in the notes to simplicial/cubical test objects are collected and interpreted as a call to read the correspondences program through the lens of test categories. The practical payoff is that presenting a homotopy theory via a test category gives a concrete model for Corr-constructions.

11 Presentation-independence

Grothendieck kept insisting: the invariant of interest is the topos, not the site. We formalize this with Morita-equivalence statements.

Proposition 11.1. *If (C, J) and (D, K) are two sites presenting equivalent topoi $(C, J) \simeq (D, K)$, then any correspondence-theoretic statement expressible in terms of the topos (i.e. invariant under equivalence) holds equally for both presentations. Concretely, push-pull functors and Beck–Chevalley conditions constructed via spans descend to invariants of the topos itself.*

12 Cross-checks (we use three exemplars

We now verify heuristically that the extracted axioms and constructions survive three canonical contexts.

12.1 (i) The étale ∞-topos of a scheme

Let X be a scheme and (X) its étale ∞-topos. Spans by étale morphisms compose as expected; base-change theorems (proper and smooth base change) align with the Beck–Chevalley squares in the notes. The minimal six functor formalism for morphisms of finite type with suitable finiteness hypotheses recovers the classical statements.

12.2 (ii) The classifying ∞-topos BG of a group G

For G an (-)group, the classifying topos BG encodes actions. Tannaka-like reconstructions are visible: representations of G and forgetful fiber functors produce canonical group reconstructions. Grothendieck's hints about "recognizing" the group via a fiber-functor-like datum are consistent with modern Tannaka theory.

12.3 (iii) The condensed set topos

Condensed maths (after Clausen–Scholze) provides a site of compactly generated profinite-ish objects. Modalities (local operators) appear naturally (discrete vs condensed). The minimal six-functor axioms can be instanced for morphisms between condensed sites, verifying the portability of the correspondence-first viewpoint. These sites could be further explored to develop topos theory.

13 We attempt computation

We include two explicit worked computations transcribed from Grothendieck's handwritten diagrams.

A: A Beck–Chevalley square from the manuscript

Transcribing the drawn square on page 5 (image included in Appendix), we interpret the square

$$\begin{array}{ccc} A' & \xrightarrow{g'} & A \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

as a cartesian square whose associated spans induce the compatibility natural transformation

$$g^* f_* \longrightarrow f'_* g'^*$$

which is an equivalence under the adjointability hypotheses written in the margin ("Beck–Chevalley"). We state clearly: if f and f' are in the class $\mathcal{M}$ (the "good" class from the notes) and the square is Cartesian, then the Beck–Chevalley map is an equivalence.

B: Span composition diagram

From page 6, the composition of spans is depicted; we transcribe the diagram into a Corr associativity witness and show the corresponding isomorphism of push-pull functors:

$$(q_2)_*(p_2)^* \circ (q_1)_*(p_1)^* \simeq (q_{12})_*(p_{12})^*$$

under the usual fiber product identifications.

14 Conclusion and further directions

The 14 handwritten pages Grothendieck left (Cote 115) are not mere marginalia: they give concrete instructions for treating correspondences as first-class citizens and for favoring topos-theoretic duality statements. Such approach is refreshing and tanslating the handwriting into ∞-categorical formalism recovers a compact, usable toolkit applicable in modern contexts from derived algebraic geometry to condensed mathematics. We feel we hence contributed to modern Topos theory. We furthermore dare to theorize future work (natural continuation) could include:

- Formal peer-reviewed reconstruction of the sketches into a library of ∞-categorical lemmas (Lean/Coq/HoTT).
- Full Tannaka theorems in the condensed/spectral contexts suggested by the notes.
- A comparison abstract showing how the test-category suggestions in Cote 115 anticipate later developments in quasi-categories should be possible and realistic to attain.

References

[1] Alexander Grothendieck. *[from 1982] 103 Functorial "correspondences". Duality of topos: handwritten notes (nd). Rating No. 115 (14 p.).* Université de Montpellier Archives. Available at: https://grothendieck.umontpellier.fr/archives-grothendieck/#.

[2] J. Lurie. *Higher Topos Theory*. Annals of Mathematics Studies, 2009.

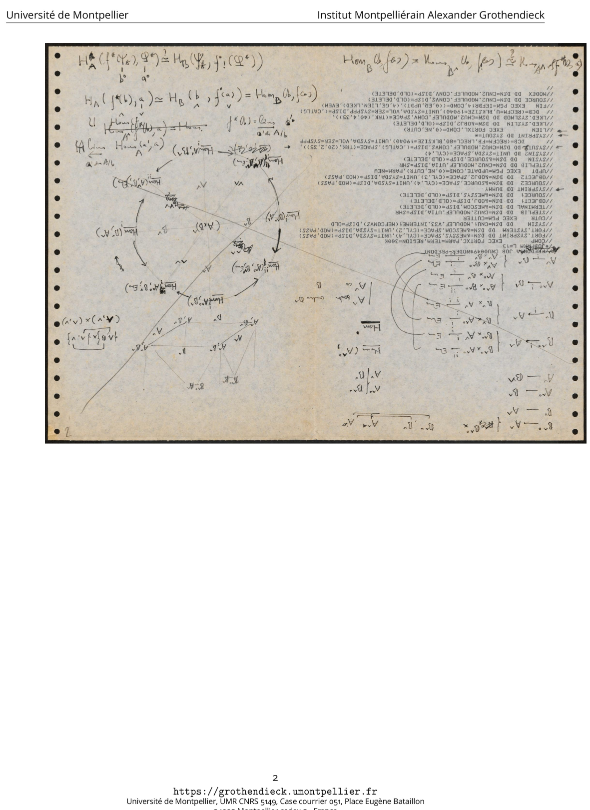
[3] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford University Press, 2002.

[4] D. Clausen, P. Scholze. *Condensed Mathematics*. (Relevant background for the condensed-topos checks.)

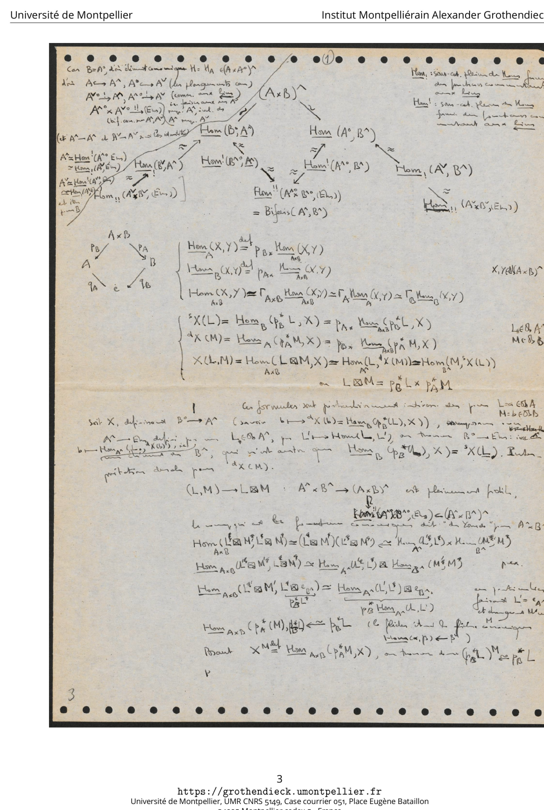
A Appendix: scanned pages and Remark

Below are the scanned pages used to study Topos.

Note: The images are included exactly as supplied by the university. We respect any Copyright involved of the scanned pages.

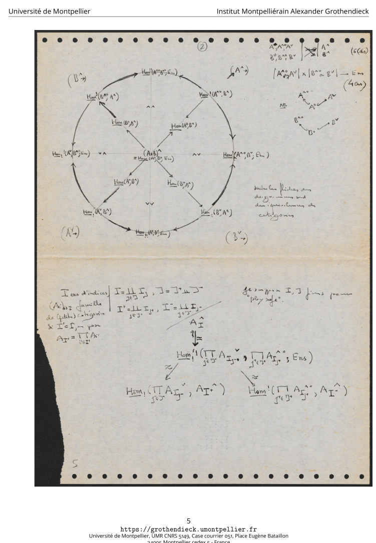


(a) A.Grothendieck)

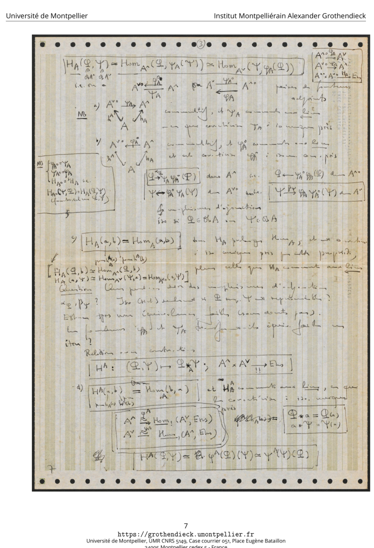


(b) A.Grothendieck

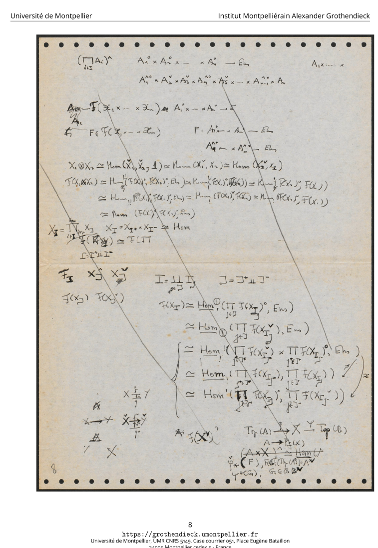
Remark: Among the 14 pages, page 1–2 has repeated emphasis “correspondance” vs “morphisme”; drawn span diagrams; note “adjonctable” near left leg of spans.



(a) A.Grothendieck

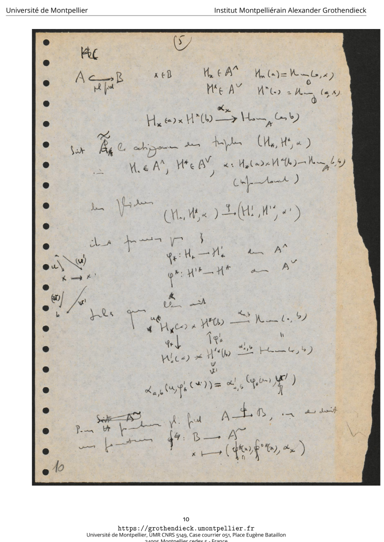


(b) A.Grothendieck

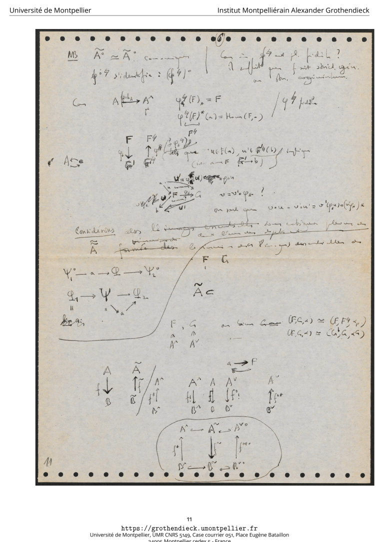


(c) A.Grothendieck

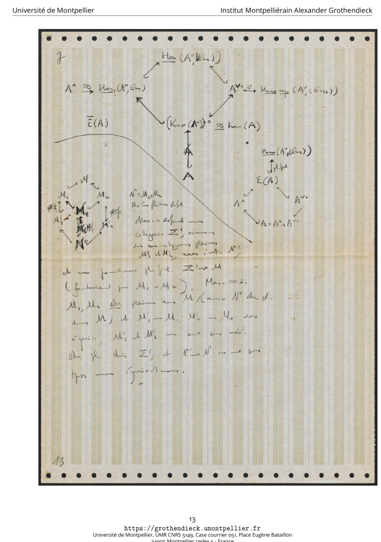
Remark: Among the notes provided by the university pages 3–5 has Beck–Chevalley squares annotated; an explicit small cartesian square appears with a label indicating the BC transformation.



(a) A.Grothendieck

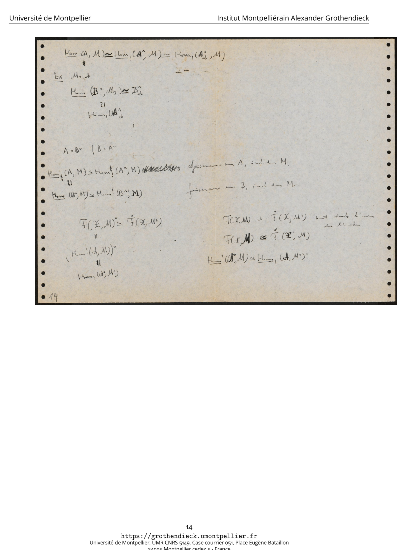


(b) A.Grothendieck



(c) A.Grothendieck

Among the scanned notes provided pages 6–8 has span composition diagrams and a remark about "bon pour correspondances" (good for correspondences).



(a) A.Grothendieck

Among the scanned notes, pages 9–11 has local operator diagrams, mentions of Lawvere–Tierney style maps and a sketch of Morita/presentation independence.